MATHEMATICS OF COMPUTATION, VOLUME 28, NUMBER 127, JULY 1974, PAGES 769-778

A Class of Quadrature Formulas*

By Ravindra Kumar

Abstract. It is proved that there exists a set of polynomials orthogonal on $[-1, 1]$ with respect to the weight function

$$w(t)/(t-x) \tag{1}$$

corresponding to the polynomials orthogonal on $[-1, 1]$ with respect to the weight function w. Simplified forms of such polynomials are obtained for the special cases

$$\begin{aligned} w(t) &= (1-t^2)^{-1/2}, \\ &= (1-t^2)^{1/2}, \\ &= ((1-t)/(1+t))^{1/2}, \end{aligned} \tag{2}$$

and the generating functions and the recurrence relation are also given. Subsequently, a set of quadrature formulas given by

$$\int_{-1}^{1} (1+t)^{p-1/2}(1-t)^{q-1/2}(1+a^2+2at)^{-1}f(t)\,dt = \sum_{k=1}^{n} H_k f(t_k) + E_n(f) \tag{3}$$

for $(p, q) = (0, 0)$, $(0, 1)$ and $(1, 1)$ is established; these formulas are valid for analytic functions. Convergence of the quadrature rules is discussed, using a technique based on the generating functions. This method appears to be simpler than the one suggested by Davis [2, pp. 311-312] and used by Chawla and Jain [3]. Finally, bounds on the error are obtained.

1. Introduction. Szegö [1] has pointed out the possible existence of orthonormal polynomials in $[-1, 1]$ corresponding to weight functions of the kind

$$w/\rho \tag{4}$$

where w is given by (2) and ρ is a polynomial satisfying certain conditions in $[-1, 1]$. A suitable choice for ρ is found to be

$$\rho(t) = 1 + a^2 + 2at \tag{4'}$$

which further suggests the existence of polynomials orthogonal on $[-1, 1]$ with respect to the weight function (1).

In this paper, a theorem is established which shows that the polynomials orthogonal on $[-1, 1]$ with regard to (1) are linear combinations of the polynomials which are orthogonal on $[-1, 1]$ with regard to w. Particular cases of w given in (2) are of special interest and they are dealt with in detail in the following sections.

Received July 7, 1972; revised November 20, 1972, May 15, 1973 and October 29, 1973.

AMS (MOS) subject classifications (1970). Primary 42A52, 65D30; Secondary 30A82.

Key words and phrases. Weight function, orthogonal polynomials, generating function, recurrence relation, quadrature formulas, convergence, bound of error.

*Part of the work was carried out at the University of Lancaster, England during the period of a visiting fellowship.

Finally, the corresponding quadrature formulas are developed and their convergence is discussed by a different method. This method, depending on the use of generating functions, is a simplification of the one used in [3]. Certain lemmas are proved which are subsequently used to find bounds on the error in formulas (3).

2. Derivation of Formulas. Let w be a fixed positive, integrable function defined on $[-1, 1]$ and let $\{\psi_n\}$ be the polynomials that are orthogonal on $[-1, 1]$ with regard to the weight function w. Then

$$\int_{-1}^{1} w(t)\psi_n(t)t^r\,dt = 0, \qquad r = 0, 1, \ldots, n-1. \tag{5}$$

We propose to find the polynomial ϕ_n of degree n in t such that

$$\int_{-1}^{1} \frac{w(t)}{t-x}\phi_n(t)t^r\,dt = 0, \qquad r = 0, 1, \ldots, n-1, \tag{6}$$

where x is a constant such that $|x| > 1$.

From (6) we have

$$\begin{aligned} &\int_{-1}^{1} w(t)\phi_n(t)\frac{t^{r+1}-xt^r}{t-x}\,dt = 0, && r = 0, 1, \ldots, n-2,\\ \Rightarrow &\int_{-1}^{1} w(t)\phi_n(t)t^r\,dt = 0, && r = 0, 1, \ldots, n-2,\\ \Rightarrow &\int_{-1}^{1} w(t)\phi_n(t)\psi_r(t)\,dt = 0, && r = 0, 1, \ldots, n-2. \end{aligned} \tag{7}$$

By expressing ϕ_n in the form $\sum_{s=0}^{n} a_s\psi_s$ and substituting in (7), we see that, since $a_n \neq 0$, we may write

$$\phi_n = \psi_n - \alpha_n\psi_{n-1} \tag{8}$$

where α_n is some constant depending on n.

Introduction of (8) in (6) with $r = 0$ and a little manipulation gives

$$\alpha_n = I_n/I_{n-1}, \tag{9}$$

$$I_n = \int_{-1}^{1} \frac{w(t)}{t-x}\psi_n(t)\,dt. \tag{10}$$

We have thus established the following result

THEOREM 1. *Given a set of polynomials* $\{\psi_n\}$ *such that*

$$\int_{-1}^{1} w(t)\psi_m(t)\psi_n(t)\,dt = 0, \qquad m \neq n,$$

there is defined a set of polynomials $\{\phi_n\}$ *given by*

$$\phi_n = \psi_n - \alpha_n\psi_{n-1}$$

such that

$$\int_{-1}^{1} \frac{w(t)}{t-x}\phi_m(t)\phi_n(t)\,dt = 0, \qquad m \neq n,$$

where

$$\alpha_n = \frac{I_n}{I_{n-1}}, \qquad I_n = \int_{-1}^{1} \frac{w(t)}{t - x} \psi_n(t)\, dt, \qquad |x| > 1.$$

The following particular cases follow from above.

3. Case I. Let $w(t) = (1 - t^2)^{-1/2}$ so that $\psi_n = T_n$ is the Chebyshev polynomial of degree n of the first kind. Let

$$x = -\tfrac{1}{2}(a + 1/a), \quad a \text{ being real},$$

so that $|x| > 1$, whatever a. With this, (10) gives

$$I_n = 2a \int_{-1}^{1} \frac{(1 - t^2)^{-1/2}}{1 + a^2 + 2at} T_n(t)\, dt. \tag{11}$$

The generating function for Chebyshev polynomials can be written as

$$\frac{1}{2} \frac{1 - w^2}{1 - 2tw + w^2} = \frac{1}{2} + \sum_{n=1}^{\infty} T_n(t) w^n, \qquad |w| < 1.$$

With $w = -1/a$, this becomes

$$\frac{1}{1 + a^2 + 2at} = \frac{2}{a^2 - 1} \left[\frac{1}{2} + \sum_{r=1}^{\infty} (-1)^r a^{-r} T_r(t) \right], \qquad |a| > 1.$$

Inserting this in (11), using the orthogonality property of the Chebyshev polynomials and the result

$$\int_{-1}^{1} (1 - t^2)^{-1/2} T_n^2(t)\, dt = \frac{\pi}{2}, \qquad n \geq 1,$$

we get

$$I_n = (-1)^n a^{-n+1} \cdot \frac{2\pi}{a^2 - 1} \quad \text{and} \quad \alpha_n = \frac{I_n}{I_{n-1}} = -\frac{1}{a}.$$

Thus, from (8), we get

$$p_n = a \cdot \phi_n = aT_n + T_{n-1}, \qquad n \geq 1, |a| > 1.$$

It is easy to prove that the corresponding orthonormal polynomials are

$$p_0^* = \left(\frac{a^2 - 1}{\pi} \right)^{1/2}, \qquad p_n^* = \left(\frac{2}{\pi} \right)^{1/2} [aT_n + T_{n-1}], \quad n \geq 1, \tag{12}$$

which satisfy the orthonormality condition

$$\int_{-1}^{1} (1 - t^2)^{-1/2} (1 + a^2 + 2at)^{-1} p_m^*(t) p_n^*(t)\, dt = \delta_{mn}, \qquad a \neq 1, m, n \geq 0, \tag{13}$$

and the recurrence relation

$$p_{n+1}^*(t) = 2t p_n^*(t) - p_{n-1}^*(t), \qquad n \geq 2. \tag{14}$$

The generating function for the Chebyshev polynomials can be written as

$$\frac{1}{2} \frac{1 - w^2}{1 - 2tw + w^2} = \sum_{n=0}^{\infty}{}' \, w^n T_n(t) = \frac{1}{2} + w \sum_{n=0}^{\infty} w^n T_{n+1}(t) = -\frac{1}{2} + \sum_{n=0}^{\infty} w^n T_n(t).$$

This gives

$$(15)\quad \begin{aligned}\frac{1}{2}\frac{1-w^2}{1-2tw+w^2}(a+w) &= a\left[\frac{1}{2}+w\sum_{n=0}^{\infty} w^n T_{n+1}(t)\right]+w\left[-\frac{1}{2}+\sum_{n=0}^{\infty} w^n T_n(t)\right]\\ &= \frac{a-w}{2}+\sum_{n=0}^{\infty} w^{n+1}[aT_{n+1}(t)+T_n(t)].\end{aligned}$$

Insertion of (12) in (15) and a little manipulation leads to the generating function (16) for the polynomials (12)

$$(16)\qquad \frac{1}{2}\frac{(1-w^2)(a+w)}{1-2tw+w^2}=\frac{a-w}{2}+\sqrt{\frac{\pi}{2}}\sum_{n=0}^{\infty} w^{n+1}p^*_{n+1}(t).$$

Polynomials (12) give rise to the quadrature formulas

$$(17)\qquad \int_{-1}^{1}(1-t^2)^{-1/2}(1+a^2+2at)^{-1}f(t)\,dt=\sum_{k=1}^{n} H_k f(t_k)+E_n(f)$$

which are exact for all polynomials of degree $\leq 2n-1$. The weight coefficients and the error term in (17) are calculated through standard methods to be given by

$$(18)\qquad H_k=-2/[p^*_{n+1}(t_k)p^{*\prime}_n(t_k)],$$

and

$$(19)\qquad E_n(f)=\frac{\pi}{(2n)!\,2^{2n-1}a^2}f^{(2n)}(\xi),\qquad -1<\xi<1,$$

where the prime denotes the derivative and $\{t_k\}$ are the zeros of the nth degree polynomial p^*_n.

4. Case II. Let $w(t)=(1-t^2)^{1/2}$ so that $\psi_n=U_n$ is the Chebyshev polynomial of degree n of the second kind. Following the procedure of Section 3, relations (12) to (14) become

$$(20)\qquad q^*_0=(2/\pi)^{1/2},\qquad q^*_n=(2/\pi)^{1/2}[aU_n+U_{n-1}],\quad n\geq 1,$$

$$(21)\qquad \int_{-1}^{1}(1-t^2)^{1/2}(1+a^2+2at)^{-1}q^*_m(t)q^*_n(t)dt=\delta_{mn},\quad a\neq 1, m,n\geq 0,$$

$$(22)\qquad q^*_{n+1}(t)=2tq^*_n(t)-q^*_{n-1}(t),\qquad n\geq 2.$$

The generating function for q^*_n can similarly be written as

$$(23)\qquad \frac{a+w}{1-2tw+w^2}=a+\sqrt{\frac{\pi}{2}}\sum_{n=0}^{\infty} w^{n+1}q^*_{n+1}(t).$$

The corresponding quadrature formula is given by

$$(24)\qquad \int_{-1}^{1}(1-t^2)^{1/2}(1+a^2+2at)^{-1}f(t)\,dt=\sum_{k=1}^{n} H_k f(t_k)+E_n(f),$$

where

$$(25)\qquad H_k=-2/[q^*_{n+1}(t_k)q^{*\prime}_n(t_k)],$$

(26) $$E_n(f) = \frac{\pi}{(2n)!\, 2^{2n+1} a^2} f^{(2n)}(\xi), \qquad -1 < \xi < 1,$$

and $\{t_k\}$ are the zeros of q_n^*.

5. Case III. With $w(t) = ((1-t)/(1+t))^{1/2}$ and orthonormal polynomials r_n^*, the corresponding results are as follows:

(27) $$r_0^* = \frac{1}{\sqrt{\pi}}(a-1), \quad r_1^*(t) = \frac{1}{\sqrt{\pi}}(2at + a + 1),$$
$$r_n^* = \frac{1}{\sqrt{\pi}}[aU_n + (1+a)U_{n-1} + U_{n-2}], \qquad n \geq 2,$$

(28) $$\int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{1/2} (1 + a^2 + 2at)^{-1} r_m^*(t) r_n^*(t)\, dt = \delta_{mn}, \qquad a \neq 1, m, n \geq 0,$$

(29) $$r_{n+1}^*(t) = 2tr_n^*(t) - r_{n-1}^*(t), \qquad n \geq 1.$$

(30) $$\frac{a + (1+a)w + w^2}{1 - 2tw + w^2} = a + 2atw + (1+a)w + (\pi)^{1/2} \sum_{n=0}^{\infty} r_{n+2}^*(t) w^{n+2}.$$

The relations corresponding to (17), (18) and (19) are

(31) $$\int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{1/2} (1 + a^2 + 2at)^{-1} f(t)\, dt = \sum_{k=1}^{n} H_k f(t_k) + E_n(f),$$

(32) $$H_k = -2/[r_{n+1}^*(t_k) r_n^{*\prime}(t_k)],$$

(33) $$E_n(f) = \frac{\pi}{(2n)!\, 2^{2n} a^2} f^{(2n)}(\xi), \qquad -1 < \xi < 1,$$

where $\{t_k\}$ are the zeros of r_n^*.

We now discuss the convergence of the quadrature rules.

6. Case I. Let L be a closed contour enclosing the interval $[-1, 1]$ in the z-plane and let the zeros of the polynomials p_n^* be denoted by $\{t_i\}_1^n$. Application of the residue theorem to the contour integral

$$\frac{1}{2\pi i} \int_L \frac{f(z)\, dz}{(z-t) p_n^*(z)}$$

gives

(34) $$f(t) = \sum_{i=1}^{n} \frac{p_n^*(t)}{(t - t_i) p_n^{*\prime}(t_i)} f(t_i) + \frac{1}{2\pi i} \int_L \frac{f(z) p_n^*(t)\, dz}{(z-t) p_n^*(z)},$$

assuming that $f(z)$ is regular within L.

Multiplying both sides of (34) with $(1-t^2)^{-1/2}(1 + a^2 + 2at)^{-1}$, integrating with regard to t on $[-1, 1]$ and interchanging the order of integration on the right-hand side, we get

(35) $$\int_{-1}^{1} \frac{f(t)\, dt}{(1-t^2)^{1/2}(1 + a^2 + 2at)} = \sum_{i=1}^{n} \mu_i f(t_i) + E_n(f)$$

where

$$\mu_i = \frac{1}{p_n^{*\prime}(t_i)} \int_{-1}^{1} \frac{p_n^*(t)\,dt}{(t - t_i)(1 - t^2)^{1/2}(1 + a^2 + 2at)} \tag{36}$$

and

$$E_n(f) = \frac{1}{2\pi i} \int_L \frac{f(z)}{p_n^*(z)} \int_{-1}^{1} \frac{p_n^*(t)\,dt}{(z - t)(1 - t^2)^{1/2}(1 + a^2 + 2at)}\,dz.$$

This is the quadrature formula (3) with $(p, q) = (0, 0)$ for analytic functions with abscissas t_i and weights μ_i.

The error of the quadrature formula can be written as

$$E_n(f) = \frac{1}{\pi i} \int_L \frac{f(z) Q_n^*(z)}{p_n^*(z)}\,dz \tag{37}$$

where

$$Q_n^*(z) = \frac{1}{2} \int_{-1}^{1} \frac{p_n^*(t)\,dt}{(1 - t^2)^{1/2}(z - t)(1 + a^2 + 2at)} \tag{38}$$

is a single-valued function for all z in the plane with the interval $[-1, 1]$ deleted.

The mapping $z = \frac{1}{2}(\xi + \xi^{-1})$, $\xi = \rho e^{i\theta}$ $(0 \le \theta \le 2\pi)$ is now introduced which maps the exterior of the unit circle $|\xi| = 1$ conformally onto the z-plane with the interval $[-1, 1]$ deleted. The circle $|\xi| = \rho$ $(\rho > 1)$ is mapped onto an ellipse ε_ρ with foci at $z = \pm 1$ and semi-axes $\frac{1}{2}(\rho + \rho^{-1})$ and $\frac{1}{2}(\rho - \rho^{-1})$.

7. A Lemma for $Q_n^*(z)$. Relation (38) with $\eta = \xi^{-1}$ now becomes

$$Q_n^*(z) = \eta \int_{-1}^{1} \frac{p_n^*(t)\,dt}{(1 - t^2)^{1/2}(1 + a^2 + 2at)(1 - 2\eta t + \eta^2)}. \tag{39}$$

Relation (16) with η for w gives

$$\frac{1}{1 - 2\eta t + \eta^2} = \frac{2}{(a + \eta)(1 - \eta^2)} \left\{ \frac{a - \eta}{2} + \sqrt{\frac{\pi}{2}} \sum_0^\infty \eta^{r+1} p_{r+1}^*(t) \right\}.$$

Inserting this in (39) and using the orthonormality property of the polynomials p_n^*, we get

$$Q_n^*(z) = \frac{\sqrt{2\pi}}{(1 - \eta^2)(a + \eta)} \eta^{n+1} = \frac{\sqrt{2\pi}}{(1 - 1/\xi^2)(a + 1/\xi)} \xi^{-n-1}.$$

Hence, for z on ϵ_ρ, we have

$$|Q_n^*(z)| \le \frac{\sqrt{2\pi}}{(1 - 1/\rho^2)(a - 1/\rho)} \rho^{-n-1} = \frac{\sqrt{2\pi}}{(\rho^2 - 1)(a\rho - 1)} \rho^{2-n}.$$

We have thus proved the following lemma.

LEMMA. *For z on ε_ρ,*

$$|Q_n^*(z)| \le \frac{\sqrt{2\pi}}{(\rho^2 - 1)(a\rho - 1)} \rho^{2-n}. \tag{40}$$

8. Convergence of the Quadrature Formula. Since, for z on ε_ρ, $T_n(z) = \frac{1}{2}(\xi^n + \xi^{-n})$, we have

$$|T_n(z)| \geq \tfrac{1}{2}(\rho^n - \rho^{-n}) \quad \text{and} \quad |T_{n-1}(z)| \leq \tfrac{1}{2}(\rho^{n-1} + \rho^{1-n}).$$

Also

$$p_n^*(z) = (2/\pi)^{1/2}[aT_n(z) + T_{n-1}(z)].$$

Therefore

$$|p_n^*(z)| \geq (2/\pi)^{1/2} \cdot \tfrac{1}{2} \cdot [a(\rho^n - \rho^{-n}) - (\rho^{n-1} + \rho^{1-n})]. \tag{41}$$

From (37), by selecting the contour as an ellipse ε_ρ ($\rho > 1$), it follows that

$$|E_n(f)| \leq \frac{1}{\pi} \int_{\varepsilon_\rho} \frac{|f(z)| \cdot |Q_n^*(z)|}{|p_n^*(z)|} ds \qquad (|dz| = ds). \tag{42}$$

Let

$$M(\rho) = \max_{z \in \epsilon_\rho} |f(z)| \quad \text{and} \quad l(\varepsilon_\rho) = \text{length of } \varepsilon_\rho. \tag{43}$$

Inserting (40), (41) and (43) in (42), we get

$$|E_n(f)| \leq \frac{2lM}{(\rho^2 - 1)(a\rho - 1)} \frac{\rho^{2-n}}{a(\rho^n - \rho^{-n}) + (\rho^{n-1} - \rho^{1-n})}.$$

Thus, the following result has been established.

THEOREM 2. *Let* $f \in A(\epsilon_\rho)$ *and let* $\rho > 1$. *Then*

$$|E_n(f)| \leq \frac{2lM}{(\rho^2 - 1)(a\rho - 1)} \frac{\rho^{2-n}}{a(\rho^n - \rho^{-n}) + (\rho^{n-1} - \rho^{1-n})}. \tag{44}$$

9. Case II. Corresponding to $(p, q) = (1, 1)$ in formula (3), relations (35) to (39) are revised as follows:

$$\int_{-1}^{1} \frac{(1 - t^2)^{1/2}}{1 + a^2 + 2at} f(t)\, dt = \sum_{i=1}^{n} \mu_i f(t_i) + E_n(f), \tag{45}$$

$$\mu_i = \frac{1}{q_n^{*\prime}(t_i)} \int_{-1}^{1} \frac{(1 - t^2)^{1/2} q_n^*(t)}{(t - t_i)(1 + a^2 + 2at)} dt, \tag{46}$$

$$E_n(f) = \frac{1}{\pi i} \int_L \frac{f(z) Q_n^*(z)}{q_n^*(z)} dz, \tag{47}$$

$$Q_n^*(z) = \frac{1}{2} \int_{-1}^{1} \frac{(1 - t^2)^{1/2} q_n^*(t)\, dt}{(z - t)(1 + a^2 + 2at)}, \tag{48}$$

$$Q_n^*(z) = \eta \int_{-1}^{1} \frac{(1 - t^2)^{1/2}}{1 + a^2 + 2at} \frac{q_n^*(t)\, dt}{1 - 2\eta t + \eta^2}, \tag{49}$$

where t_i are the zeros of q_n^*.

Inserting (23) with η for w in (49) and using the orthonormality property of the polynomials q_n^*, we get

$$Q_n^*(z) = \sqrt{\frac{\pi}{2}}\,\frac{\eta^{n+1}}{a+\eta} = \sqrt{\frac{\pi}{2}}\,\frac{\xi^{-n-1}}{a+1/\xi}$$

which proves the following lemma.

LEMMA. *For z on ε_ρ,*

$$|Q_n^*(z)| \le \sqrt{\frac{\pi}{2}}\,\frac{\rho^{-n}}{a\rho - 1}. \tag{50}$$

10. Bounds on Error. Since

$$|z_1 - z_2| \ge ||z_1|-|z_2|| \quad \text{and} \quad q_n^*(z) = (2/\pi)^{1/2}[aU_n(z) + U_{n-1}(z)]$$

we have

$$|q_n^*(z)| \ge (2/\pi)^{1/2}[a|U_n(z)| - |U_{n-1}(z)|].$$

Now, for z on ε_ρ,

$$U_n(z) = (\xi^{n+1} - \xi^{-n-1})/(\xi - \xi^{-1}).$$

Therefore

$$|U_n(z)| \ge \frac{\rho^{n+1} - \rho^{-n-1}}{\rho + \rho^{-1}} \quad \text{and} \quad |U_{n-1}(z)| \le \frac{\rho^n + \rho^{-n}}{\rho - \rho^{-1}}.$$

Hence

$$|q_n^*(z)| \ge \left(\frac{2}{\pi}\right)^{1/2}\left[a\frac{\rho^{n+1} - \rho^{-n-1}}{\rho + \rho^{-1}} - \frac{\rho^n + \rho^{-n}}{\rho - \rho^{-1}}\right]. \tag{51}$$

From (47), we have

$$|E_n(f)| \le \frac{1}{\pi}\int_{\varepsilon_\rho} \frac{|f(z)|\,|Q_n^*(z)|}{|q_n^*(z)|}\,ds \qquad (|dz| = ds). \tag{52}$$

Inserting (50), (51) and (43) in (52), we get, on simplification, the following result:

THEOREM 3. *Let $f \in A(\epsilon_\rho)$ and let $\rho > 1$. Then*

$$|E_n(f)| \le \frac{M(\rho)l(\varepsilon_\rho)}{2}\,\frac{\rho^{-n}}{a\rho^{-1}}\cdot\left(a\left(\frac{\rho^{n+1} - \rho^{-n-1}}{\rho + \rho^{-1}}\right) - \left(\frac{\rho^n + \rho^{-n}}{\rho - \rho^{-1}}\right)\right)^{-1} \tag{53}$$

where $M(\rho)$ and $l(\varepsilon_\rho)$ are given by (43).

11. Case III. Corresponding to $(p, q) = (0, 1)$ in formula (3), relations (35) to (39) are revised as follows:

$$\int_{-1}^{1}\left(\frac{1-t}{1+t}\right)^{1/2}(1 + a^2 + 2at)^{-1}f(t)\,dt = \sum_{i=1}^{n}\mu_i f(t_i) + E_n(f), \tag{54}$$

$$\mu_i = \frac{1}{r_n^{*\prime}(t_i)}\int_{-1}^{1}\left(\frac{1-t}{1+t}\right)^{1/2}\frac{r_n^*(t)\,dt}{(t - t_i)(1 + a^2 + 2at)}, \tag{55}$$

(56) $$E_n(f) = \frac{1}{\pi i}\int_L \frac{f(z)Q_n^*(z)}{r_n^*(z)}dz,$$

(57) $$Q_n^*(z) = \frac{1}{2}\int_{-1}^{1}\left(\frac{1-t}{1+t}\right)^{1/2}\frac{r_n^*(t)}{(z-t)(1+a^2+2at)}dt,$$

where t_i are the zeros of $r_n^*(t)$,

(58) $$Q_n^*(z) = \eta\int_{-1}^{1}\left(\frac{1-t}{1+t}\right)^{1/2}\frac{r_n^*(t)}{1+a^2+2at}\frac{dt}{1-2\eta t+\eta^2}.$$

Introduction of (30) in (58), with η for w, and the use of orthonormality property of the polynomials r_n^*, we get

$$Q_n^*(z) = (\pi)^{1/2}\frac{\eta^{n+1}}{a+(1+a)\eta+\eta^2} = (\pi)^{1/2}\frac{\xi^{-n+1}}{a\xi^2+(1+a)\xi+1},$$

which proves the following lemma:

LEMMA. *For z on ε_ρ,*

(59) $$|Q_n^*(z)| \le (\pi)^{1/2}\frac{\rho^{-n+1}}{a\rho^2-(1+a)\rho+1} \qquad (n > 1).$$

12. Bounds on Error. Since

$$|z_1+z_2| \ge ||z_1|-|z_2|| \quad \text{and} \quad r_n^*(z) = (\pi)^{-1/2}[aU_n(z) + \{(1+a)U_{n-1}(z)+U_{n-2}(z)\}]$$

we have

$$|r_n^*(z)| \ge (\pi)^{-1/2}[a|U_n(z)| - \{(1+a)|U_{n-1}(z)| + |U_{n-2}(z)|\}].$$

Now, for z on ε_ρ,

$$U_n(z) = (\xi^{n+1}-\xi^{-n-1})/(\xi-\xi^{-1}).$$

Therefore

$$|U_n(z)| \ge \frac{\rho^{n+1}-\rho^{-n-1}}{\rho+\rho^{-1}},$$

$$|U_{n-1}(z)| \le \frac{\rho^n+\rho^{-n}}{\rho-\rho^{-1}} \quad \text{and} \quad |U_{n-2}(z)| \le \frac{\rho^{n-1}+\rho^{-n+1}}{\rho-\rho^{-1}}.$$

Hence

(60) $$|r_n^*(z)| \ge (\pi)^{-1/2}\left[a\frac{\rho^{n+1}-\rho^{-n-1}}{\rho+\rho^{-1}} - (1+a)\frac{\rho^n+\rho^{-n}}{\rho-\rho^{-1}} - \frac{\rho^{n-1}+\rho^{-n+1}}{\rho-\rho^{-1}}\right].$$

From (56) we have

(61) $$|E_n(f)| \le \frac{1}{\pi}\int_{\varepsilon_\rho}\frac{|f(z)|\cdot|Q_n^*(z)|}{|r_n^*(z)|}ds \qquad (|dz| = ds).$$

Inserting (59), (60) and (43) in (61), we get, on simplification, the following result:

THEOREM 4. *Let* $f \in A(\epsilon_\rho)$ *and let* $\rho > 1$. *Then*

$$|E_n(f)| \le \frac{M(\rho)l(\varepsilon_\rho)\rho^{-n+1}}{a\rho^2 - (1+a)\rho + 1} \cdot \left(a\left(\frac{\rho^{n+1} - \rho^{-n-1}}{\rho + \rho^{-1}}\right) - (1+a)\left(\frac{\rho^n + \rho^{-n}}{\rho - \rho^{-1}}\right) - \left(\frac{\rho^{n-1} + \rho^{-n+1}}{\rho - \rho^{-1}}\right)\right)^{-1}. \tag{62}$$

13. Acknowledgements. Thanks are due to Professor M. K. Jain for encouragement and inspiration.

I am grateful to the referee for his important suggestions which improved the text and presentation of the material to the present form.

I owe sincere thanks to Professor C. W. Clenshaw and to Dr. D. Kershaw for their very useful comments and helpful discussions.

Department of Mathematics
Indian Institute of Technology
Hauz Khas, New Delhi-29, India

1. G. SZEGÖ, *Orthogonal Polynomials*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, R.I., 1959.

2. P. J. DAVIS, *Interpolation and Approximation*, Blaisdell, New York, 1963. MR **28** #5160.

3. M. M. CHAWLA & M. K. JAIN, "Error estimates for Gauss quadrature formulas for analytic functions," *Math. Comp.*, v. 22, 1968, pp. 82-90. MR **36** #6142.